REMARKS ON VARIATIONAL BOUNDS FOR WAVEGUIDE SCATTERING

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Abstract

Utilization of variational techniques has facilitated the solution of scattering problems. A variational principle which gives a bound on the scattering elements in waveguide problems is presented. The formalism is applicable to scalar and tensor type problems for single and multimode scattering. A variation-iteration method can be used to avoid the need for explicitly evaluating a Green's function that appears in the formalism. Also, an alternative variational bound principle is outlined.

Introduction

The solution to scattering problems involving dielectric obstacles in waveguides has been facilitated by the use of variational techniques first developed by Schwinger and others.[1] For certain waveguide problems the variational expressions of Schwinger and others yield upper or lower bounds on the network parameters. However, the bound is not realized for truly three-dimensional problems, that is, where the fields cannot be derived from a single scalar potential. An example is a three-dimensional obstacle which touches only one waveguide surface. A method originally developed by Kato [2] for quantum mechanical potential scattering, and subsequently extended and applied to the scattering of electromagnetic waves by isotropic obstacles in a waveguide,[3] does give variational bounds on the scattering parameters. However, the method is limited to relatively simple problems because it requires that there be some solvable problems related in a rather restricted fashion to the original problem. The advantage of the Kato formalism is that, where applicable, it can provide both bounds on the scattering parameters under consideration. On the other hand, the variational bound (VB) principle [4] described here gives bounds on the quantities of interest in more general cases. The VB was applied to single-mode scattering caused by obstacles symmetric with respect to a plane perpendicular to the axis of the waveguide.[5] The technique was readily extended to the case of non-symmetric obstacles in single-mode waveguides [6] and then generalized to multimode waveguide problems.[7] One should note the distinction between bounds and variational bounds. Variational bounds contain variational parameters while bounds do not. The presence of variational parameters enables one to approach monotonically the correct results.

A basic difficulty in using the VB method is the presence of a Green's function in the formalism. However, a variation-iteration technique of quantum mechanics [8] is extended by us to waveguide problems. This method avoids the need of evaluating explicitly any Green's function and therefore facilitates the solving of more involved waveguide problems. In addition, some ideas about an alternative variational technique are discussed.

Variational Bound Principle

In this section we outline the salient points in the derivation of the VB principle for the scattering of an electromagnetic wave by an obstacle in a waveguide. On assuming a time dependence exp $(-i\omega t)$, the electric-field intensity $\vec{E}$ satisfies the equations

$$-\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) + (\varepsilon\omega^2/c^2)\vec{E} = 0 \qquad (1a)$$

$$\vec{\nabla}\cdot\varepsilon\vec{E} = 0 \qquad (1b)$$

where c is the velocity of light, ε is the relative permittivity, ω is the angular freuqency, and μ of the obstacle is 1. Equation (1a) can be rewritten as

$$(W - H)\vec{E} \equiv (W - T - V)\vec{E} = 0 \qquad (2a)$$

where

$$V \equiv \vec{\nabla}\vec{\nabla}\cdot + (1-\varepsilon)\omega^2/c^2$$

$$W \equiv \omega^2/c^2, \quad T \equiv -\nabla^2 \qquad (2b)$$

With H, W, T, and V symbolically identified with the Hamiltonian and with the total, the kinetic, and the potential energy, respectively, we have a connection with the Schroedinger equation. This purely formal connection simplifies the adaptation of the VB formalism, developed for quantum mechanical problems, to the waveguide case.

Letting the z coordinate be parallel to the axis of the waveguide, the even and odd standing wave solutions of $\vec{E}$ have the asymptotic forms [3] for $z \to \infty$:

$$\vec{E}_e = \vec{e}(x,y)[-\sin(kz+\theta) + \cot(\eta_e - \theta)\cos(kz+\theta)] \quad (3a)$$

$$\vec{E}_o = \vec{e}(x,y)[\cos(kz+\theta) + \cot(\eta_o - \theta)\sin(kz+\theta)], \quad (3b)$$

where η_e and η_o are the even and odd phase shifts, respectively, and $\vec{e}(x,y)$ is the form function for the propagating mode. With the help of (1) and (3) one arrives at the following identity:[3]

$$2\,[k \cot(\eta - \theta) - k \cot(\eta_t - \theta)] =$$

$$\int \vec{E}_t\cdot(H - W)\vec{E}_t d\tau - \int \vec{\Omega}\cdot(H - W)\vec{\Omega} d\tau \qquad (4)$$

where $\vec{E}_t$ is a trial electric field with the boundary conditions given by (3) but with η replaced by η_t,

$$\vec{\Omega} \equiv \vec{E}_t - \vec{E},$$

and the range of integration $d\tau$ is over the interior of the waveguide. On dropping the unknown second order term

$$\int \vec{\Omega}\cdot(H - W)\vec{\Omega} d\tau \qquad (5)$$

one has a variational principle. To derive a VB principle, an explicit bound on (5) must be obtained.

The spectrum of H - W for the corresponding class of functions contains a positive continous spectrum as well as a continuum of negative eigenvalues ranging from $-(W-\pi^2/a^2)$ to 0. It is possible to eliminate the positive continuum and a finite number of negative eigenvalues. But one does not know how to eliminate a continuum of negative eigenvalues, and therefore a bound on the second order term is not readily obtained. However, the replacement of the identity (4) by another identity, obtained with the use of projection operators, enables us to formulate a practical variational bound.

We define two dyadic projection operators P and Q, such that for any vector function $\vec{F}(x,y,z)$

$$P\vec{F}(x,y,z) = \vec{e}(x,y)\int \vec{e}(x',y')\cdot\vec{F}(x',y',z)dx'dy' \div \int \vec{e}(x,y)\cdot \vec{e}(x,y)\, dxdy \qquad (6a)$$

and

$$Q = 1 - P; \qquad (6b)$$

that is, P projects onto the propagating mode and Q projects onto the higher modes. The presence of Q eliminates the negative continuum contribution to the eigenvalue spectrum and therefore the resulting modified second order term can be bounded.[4] The VB principle takes the form [5]

$$2k[\cot(\eta - \theta) - \cot(\eta^P - \theta)] \le 2\int P\vec{E}^P\cdot HQ\vec{E}_t d\tau + \int Q\vec{E}_t\cdot[H + HPG^P PH - W]\, Q\vec{E}_t d\tau, \qquad (7)$$

where $P\vec{E}^P$ is the regular solution of the (so-called) static one-dimensional equation in z.

$$P(H - W)P\vec{E}^P = 0, \qquad (8)$$

and the dyadic Green's function G^P is defined by

$$P(H - W)PG^P = -P. \qquad (9)$$

An expression for multimode scattering equivalent to (7) has been derived.[6]

<u>Variation - Iteration Method</u>

The construction of the Green's functions are the most serious complications of the VB method. This can be avoided by using an iteration procedure.[8] In the derivation of the VB principle (7) (with the aid of the projection operators), (1a) was transformed into a pair of coupled equations

$$P(H - W)P\vec{E} = - PHQ\vec{E} \qquad (10)$$

$$Q(H - W)Q\vec{E} = - QHP\vec{E}. \qquad (11)$$

The Green's function G^P arises in the formal solution of (10) as

$$P\vec{E} = P\vec{E}^P + PG^P PHQ\vec{E}. \qquad (12)$$

The variation iteration method avoids the explicit evaluation of G^P. The procedure may be illustrated as follows. Choose $Q\vec{E}_{ot} = 0$ and obtain $P\vec{E}_{ot}$ by solving (10). Substitute $P\vec{E}_{ot}$ in the right hand side of (11) and solve for $Q\vec{E}_{1t}$ variationally. Next put this $Q\vec{E}_{1t}$ back into (10) and obtain $P\vec{E}_{1t}$ exactly. It can be shown from the VB principle (7) that this procedure provides a bound at every stage of the iteration. The first iteration gives

$$2k[\cot(\eta - \theta) - \cot(\eta^P - \theta)] \le \int Q\vec{E}_{1t} \cdot HP\vec{E}_{1t}\, d\tau. \qquad (13)$$

After the nth iteration we have,

$$2k[\cot(\eta - \theta) - \cot(\eta^P - \theta)] \le \int P\vec{E}^P.\ PHQ\vec{E}_{nt}\, d\tau + \int Q\vec{E}_{nt} \cdot QHP\, [P\vec{E}_{nt} - P\vec{E}_{n-1,t}]\, d\tau. \qquad (14)$$

(Note that when n=1, $P\vec{E}_{n-1,t} = P\vec{E}^P$.)

This method is useful if the interation converges rapidly.

<u>Variational Bound Without Projection Operator</u>

It is possible to obtain a VB principle without the use of projection operators. The starting point is again the identity (4). It is shown that [9]

$$\int \vec{\Omega}\cdot(H-W)\vec{\Omega}\, d\tau = \iint (H-W)\vec{E}_t\cdot G(x,y,z;x',y',z')\cdot(H-W)\vec{E}_t d\tau d\tau' \qquad (15)$$

where G satisfies

$$(H-W)G = - \vec{I}\delta(x-x')\delta(y-y')\delta(z-z') \qquad (16)$$

and where $\vec{I}$ is the idemfactor. G can be written as

$$G=G_o+G_oVG_o+G_oVG_oVG_o+\cdots = G_o+G_oVG, \qquad (17)$$

where G_o is the free (absence of obstacle), Green's function. From (17)

$$G_0 = (1-G_0V)G.$$

If $G_0V \le a < 1$ then,

$$|G| \le \left|\frac{G_0}{1-a}\right| \equiv M(x,y,z;\ x',y',z').$$

Therefore,

$$\int \vec{\Omega}\cdot(H-W)\vec{\Omega}d\tau \le \int\int |(H-W)\vec{E}_t|\ M\ |(H-W)\vec{E}_t|\,d\tau d\tau'. \quad (18)$$

If $M = N(x,y,z)\ N(n',y',z')$ the right hand side of (18) becomes $[\int|N|\ |(H-W)\vec{E}_t|\,d\tau]^2$. With the aid of (18), (4) yields bounds on the scattering parameters.

Conclusion

The VB principle has been presented for the determination of variational bounds on the elements that characterize the scattering of electromagnetic waves by dielectric obstacles in single and multimode waveguides. A modified form of the VB method (based on an iteration procedure) which avoids the need of explicitly finding the Green's function has been described. Finally a different approach for obtaining a VB principle, which has not been fully investigated, is mentioned.

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